CC2,
Continuous Charge Approach continued

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Problem

1st lasing at TTF in Feb. 2000

- Beam dynamics with CSR for FEL-BC-systems since about a **quarter century**.
- **Validity** of models is confirmed to some extend by measurements.
- **Experimental verification** is limited to currently measurable effects and realizable parameter ranges.

new developments depend on being able to access that are ranges beyond that

- **Computational verification** of models is incomplete or missing.

What is difficult?	Simplifications used today:
non linear trajectory	SC model (CUM) and/or CSR model (1d)
variable bunch shape	"1d" model with one-bend- or, multi-bend-interaction
effects by glue physics	line charge without retardation (rigid in history)
chamber	none or infinite flat (two mirrors)
surface effects:	none
simple conductivity	
anomalous skin effect	
surface layers	
surface roughness	

The Continuous Charge approach is an attempt to understand the fields of charge distributions in realistic motion in free space.

Approach

Solve the driven problem in free space, still with simplifications.

These simplifications are:
2D = motion of "source particles" in XY plane,
continuous source distribution (continuous 4D phase space),
linear optics approximation for beam dynamics,
only **magnetic lattice**,
no hard edges but **fringe fields**.
Beam dynamics with self effects is calculated in **perturbation theory**.
Restriction: **gaussian** source distribution

Perturbation approach:
Calculate phase space distribution without self effects. (Solve EOM.)
Calculate source terms of EM problem. (Unperturbed motion.)
Solve EM problem for these source terms.
Motion: treat self effects like external fields.
Sample these fields at the coordinates (location and velocity) of the unperturbed motion.

flat

Accelerator vs Cartesian Coordinates

The external magnetic field $\mathbf{B}(X,Y,Z) \to \mathbf{B}(X,Y,0) = \mathbf{e}_Z B_Z(X,Y)$
this field is smooth (there are fringe fields)

energy and initial conditions of a reference particle

$$\mathcal{E}_\mathrm{r}, X_{(\mathrm{i})}, Y_{(\mathrm{i})}, Z_{(\mathrm{i})} = 0, X'_{(\mathrm{i})}, Y'_{(\mathrm{i})}, Z'_{(\mathrm{i})} = 0 \text{ at time } t_{(\mathrm{i})}$$

$\to$ path length $S = \left(t - t_{(\mathrm{i})}\right) v_\mathrm{r}$ and reference trajectory $X_\mathrm{r}(S), Y_\mathrm{r}(S)$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X_\mathrm{r}(S) \\ Y_\mathrm{r}(S) \end{bmatrix} + \begin{bmatrix} -Y'_\mathrm{r}(S) \\ X'_\mathrm{r}(S) \end{bmatrix} x$$

$$t = \frac{S - s}{v_\mathrm{r}}$$

$x, s, S \leftrightarrow X, Y, t$

$\to$ explicit

$\leftarrow$ implicit

$z = Z$ is the same in both systems

external magnetic field in accelerator coordinates

$$B(S,x) = B_Z(X,Y) \approx B(S,0) + x\partial_x B(S,0) = \frac{p_\mathrm{r}}{q}\left(-\varphi'(S) + xK(S)\right)$$

linear field approximation

Linear Optics

decision on the state vector $\mathcal{X} = (x, x', s, \eta)^t$ with $x' = dx/dS$

$\eta = (\mathcal{E} - \mathcal{E}_r)/\mathcal{E}_r$

exact equation of motion

$$\frac{d}{dS}\mathcal{X} = f(S,\mathcal{X}) = \underbrace{f_{(e)}(S,\mathcal{X})}_{\approx M(S)\mathcal{X}} + f_{(s)}(S,\mathcal{X})\,F_{(s)}$$

Lorentz force
by self fields

linear approximation without self effects

$$M(S) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -K - \varphi'^2 & 0 & 0 & -\varphi'/\beta_r^2 \\ \varphi' & 0 & 0 & 1/(\gamma_r\beta_r)^2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

transport matrix $\mathcal{X}(S) = T(S \leftarrow A)\,\mathcal{X}(A)$

$$\frac{d}{dS}\mathcal{X}(S) = \frac{dT(S \leftarrow A)}{dS}\left(T(S \leftarrow A)\right)^{-1}\mathcal{X}(S) = M(S)\,\mathcal{X}(S)$$

solve

$$\frac{dT(S \leftarrow A)}{dS} = T(S \leftarrow A)\,M(S)$$

Density Functions

normalized gaussian 4d density $$g_4(\mathcal{X}, D) = \sqrt{\frac{\det D}{(2\pi)^4}} \exp\left(-\frac{1}{2}\mathcal{X}^t D \mathcal{X}\right)$$

with $D = C^{-1}$

the inverse of the 4D correlation matrix

initial gaussian 4d density $$f(\mathcal{X}, S=0) = q_{\text{tot}} g_4\left(\mathcal{X} - \mathcal{X}_{oi}, C_i^{-1}\right)$$

initial offset and initial 4D correlation matrix

gaussian 4d density $$f(\mathcal{X}, S) = q_{\text{tot}} g_4\left(\mathcal{X} - \mathcal{X}_o(S), (C(S))^{-1}\right)$$

with $$\mathcal{X}_{\text{o}}(S) = T(S \leftarrow S_{\text{i}})\mathcal{X}_{\text{oi}}$$
$$C(S) = T(S \leftarrow S_{\text{i}}) C_{\text{i}} T(S_{\text{i}} \leftarrow S)$$

2d functions:

projection to space
longitudinal 2d current density $$J_S(x, s, S) = v_{\text{r}} \int f\left([x, x', s, \eta]^t, S\right) dx' d\eta$$

averaged velocity $$\overline{\mathbf{v}}(x, s, S) = \frac{\int \mathbf{v}(\mathcal{X}, S) f(\mathcal{X}, S) dx' d\eta}{\int f(\mathcal{X}, S) dx' d\eta}$$

can be integrated **analytically**

Tabulated and Analytic Functions

tabulated versus S: $X_r(S),\ Y_r(S),\ \varphi(S),\ \varphi'(S),\ K(S), T(S \leftarrow A)$

these functions are calculated once, with high accuracy and good resolution

analytic: $J_S(x,s,S) = J_S(x,s,S,T(S \leftarrow A))$

$$\overline{\mathbf{v}}(x,s,S) = \overline{\mathbf{v}}(x,s,S,T(S \leftarrow A))$$

$\rightarrow$ analytic derivatives

$$\frac{\partial J_S(x,s,S)}{\partial x} = \frac{\partial J_S(x,s,S,T(S \leftarrow A))}{\partial x}$$

$$\frac{\partial J_S(x,s,S)}{\partial S} = \frac{\partial J_S(x,s,S,T(S \leftarrow A))}{\partial S} + \sum_{ij} \frac{\partial J_S(x,s,S,T(S \leftarrow A))}{\partial T_{ij}} \frac{dT_{ij}(S \leftarrow A)}{dS}$$

also known are $\dfrac{dT_{ij}(S \leftarrow A)}{dS} = \left(T(S \leftarrow A)\, M(S)\right)_{ij}$

$$J = \left(\frac{\partial[X,Y,t]}{\partial[x,s,S]} \right)$$ Jacobi matrix

We can calculate all derivatives of Js and v
(to cartesian or accelerator coordinates) with help of tabulated functions!

EM Problem: Retarded Potentials

$$\rho(x,s,z,S) = \frac{J_S(x,s,S)}{\overline{\mathbf{v}}(x,s,S)\cdot\mathbf{e}_S(S)}\delta(z)$$

$$\mathbf{J}(x,s,z,S) = \overline{\mathbf{v}}(x,s,S)\,\rho(x,s,z,S)$$

with $\mathbf{e}_S(S) = X_r'(S)\mathbf{e}_X + Y_r'(S)\mathbf{e}_Y$

$\mathbf{e}_x(S) = \mathbf{e}_X \times \mathbf{e}_S(S)$

only the flat part is needed

$$\rho(x,s,S) = \frac{J_S(x,s,S)}{\overline{\mathbf{v}}(x,s,S)\cdot\mathbf{e}_S(S)}$$

$$\mathbf{J}(x,s,S) = \overline{\mathbf{v}}(x,s,S)\,\rho(x,s,S)$$

$$V(X,Y,Z,t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho\left(x\left(\hat{X},\hat{Y},\hat{t}\right),s(\cdots),S(\cdots)\right)}{\sqrt{\left(X-\hat{X}\right)^2+\left(Y-\hat{Y}\right)^2+Z^2}}\,d\hat{X}d\hat{Y}$$

$$\mathbf{A}(X,Y,Z,t) = \frac{\mu_0}{4\pi\varepsilon}\int \frac{\mathbf{J}\left(x\left(\hat{X},\hat{Y},\hat{t}\right),s(\cdots),S(\cdots)\right)}{\sqrt{\left(X-\hat{X}\right)^2+\left(Y-\hat{Y}\right)^2+Z^2}}\,d\hat{X}d\hat{Y}$$

$\hat{t} = t - c^{-1}\sqrt{\cdots}$

retarded time

but $\hat{X},\hat{Y},\hat{t} \to x,s,S$ is implicit and expensive

Integration via Accelerator Coordinates

eliminate s: $\hat{t} = \frac{S-s}{v_r} \rightarrow s\left(\hat{X},\hat{Y},\hat{t}\right) = S - v_r\left(t + c^{-1}\sqrt{\cdots}\right) = \hat{s}$

substitution: $\hat{X} = X\left(S,x\right)$ is explicit

$$\hat{Y} = Y\left(S,x\right)$$

$$d\hat{X}d\hat{Y} = \left(1 - x\varphi'\right)dxdS$$

$$\sqrt{\cdots} = \sqrt{\left(X - X\left(S,x\right)\right)^2 + \left(Y - Y\left(S,x\right)\right)^2 + Z^2} = \sqrt{a^2\left(S\right) + \left(x - b\left(S\right)\right)^2}$$

$$V\left(X,Y,Z,t\right) = \frac{1}{4\pi\varepsilon_0}\int \rho\left(x,\hat{s},S\right)\frac{1 - x\varphi'}{\sqrt{\cdots}}dxdS$$

$$\mathbf{A}\left(X,Y,Z,t\right) = \frac{\mu_0}{4\pi\varepsilon}\int \mathbf{J}\left(x,\hat{s},S\right)\frac{1 - x\varphi'}{\sqrt{\cdots}}dxdS$$

derivatives to X, Y or t: for instance $\partial_X V$

$$\frac{\partial V\left(X,Y,Z,t\right)}{\partial X} = \frac{1}{4\pi\varepsilon_0}\int \left.\frac{\partial\rho}{\partial X}\right|_{\left(x,\hat{s},S\right)} \frac{1 - x\varphi'}{\sqrt{\cdots}}dxdS$$

... is known

as $\left(\nabla^2 - c^{-2}\partial_t^2\right)\left(\partial_X V\right) = \partial_X\left(\nabla^2 - c^{-2}\partial_t^2\right)V = \partial_X\left(\varepsilon_0^{-1}\rho\right)$

Numerical Realization

simultaneous integration of all potentials and all derivatives

$$\begin{bmatrix} V(X,Y,Z,t) \\ \mathbf{A}(X,Y,Z,t) \\ \partial_X V(X,Y,Z,t) \\ \vdots \end{bmatrix} = \frac{1}{4\pi}\int Q(x,\hat{s},S)\frac{1-x\varphi'}{\sqrt{a^2(S)+(x-b(S))^2}}dxdS$$

$$\text{with} \quad Q(x,s,S) = \begin{bmatrix} \varepsilon_0^{-1}\rho \\ \mu_0\mathbf{J} \\ \varepsilon_0^{-1}\partial_X\rho \\ \vdots \end{bmatrix}_{(x,s,S)}$$

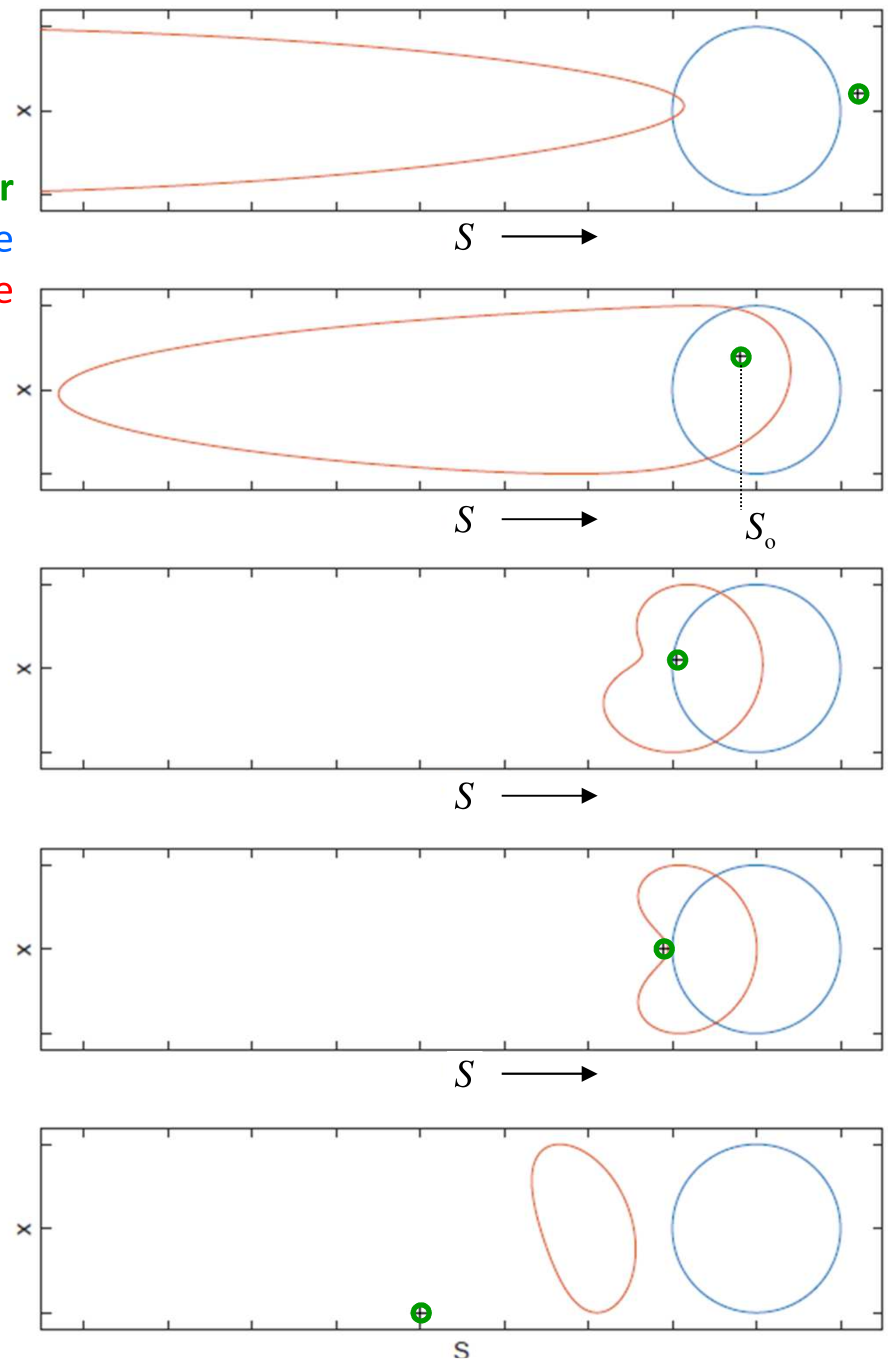
$$\rightarrow \mathbf{E} = -\nabla V - \partial_t\mathbf{A},\ \mathbf{B} = \nabla\times\mathbf{A}$$

- needs a 2d integration for each X,Y,Z,t point
- independent (parallel) computation of different points
- it is advantageous to start with the x integration
  - step width control is fairly simple
  - singularity ($a(S)\rightarrow 0$) needs some care
  - analytic approximation for large $\sqrt{\ldots}$ possible
- outer (S) integration is difficult

Sx Integration Range

example: uniform motion with $\beta = 0.9$

observer
actual shape
retarded shape



S integration is difficult:

range for $S < S_o$ is stretched
range for $S > S_o$ is compressed
both parts might be of similar importance
integral for $S \to S_o$ might be singular

extreme stretch possible:

long compared to bunch length
fringe fields
element lengths
drift lengths
section length
beam line

S Integration with Step Width Control = global & component wise

- working horse (method for subintervals from A to B)

$$I = \int_A^B f(x)\,dx \approx I_{\text{est}} = \sum_{n=1}^{N} a_n f(x_n)$$

$$\Delta I = I - I_{\text{est}} \approx \sum_{n=1}^{N} b_n f(x_n)$$

Gauss-Kronrod quadrature, moderate N
applicable to integrals that can be improper at the boundary

- clever splitting of integration range into initial subintervals
  - f.i. near-interval, far-interval, fringe-intervals, bend-interval etc
  - individual interval properties as
    integration method
    substitution of integration parameter
    error weight

- adaptive integration by splitting of intervals
  - inheritance of interval properties
  - error estimation ($\rightarrow$ working horse)
  - error criterion and refinement strategy
    global considerations
    component considerations
    noise
  - abort criterion

Tracking with Perturbation

$$\frac{d}{dS}\mathcal{X} = f(S,\mathcal{X}) = \underbrace{f_{(\mathrm{e})}(S,\mathcal{X})}_{\approx M(S)\mathcal{X}} + \underbrace{f_{(\mathrm{s})}(S,\mathcal{X})}_{\approx Q(S)+N(S)\mathcal{X}} \quad F_{(s)}(S,\mathcal{X})$$

solve linearized equation of motion

$\mathcal{X}_{\mathrm{o}}(A) = \mathcal{X}_{\mathrm{oi}}$ initial condition (per particle)

$\mathcal{X}_{\mathrm{o}}(S) = T(S \leftarrow A)\mathcal{X}_{\mathrm{oi}}$ unperturbed motion → EM sources

$\frac{d}{dS}\mathcal{X} \approx M(S)\mathcal{X} + \left(Q(S) + N(S)\mathcal{X}_{\mathrm{o}}\right) F_{(s)}(S,\mathcal{X}_{\mathrm{o}})$ perturbed motion

↓

calculate beam dynamics properties

as projected emittance, slice emittance, slice energy spread

Example

several setups have been calculated:

- 4 magnet BC chicane
  - **Zeuthen** benchmark BC, 500 MeV, 511 MeV, **5 GeV**, 5.11 GeV
  - XFEL CW BC2
  - Flash2020 BC1, BC2
- T20 beamline with combinded function magnet (XFEL)
- undulator (Pitz)

1) calculation of EM quantities:

retarded potentials and derivatives have been calculated on a xsS-grid
typically 100 .. 1000 points in S direction (along structure)
typically 500 .. 1000 xs-gridpoints (on smallest rectangle around 4σ ellipse)

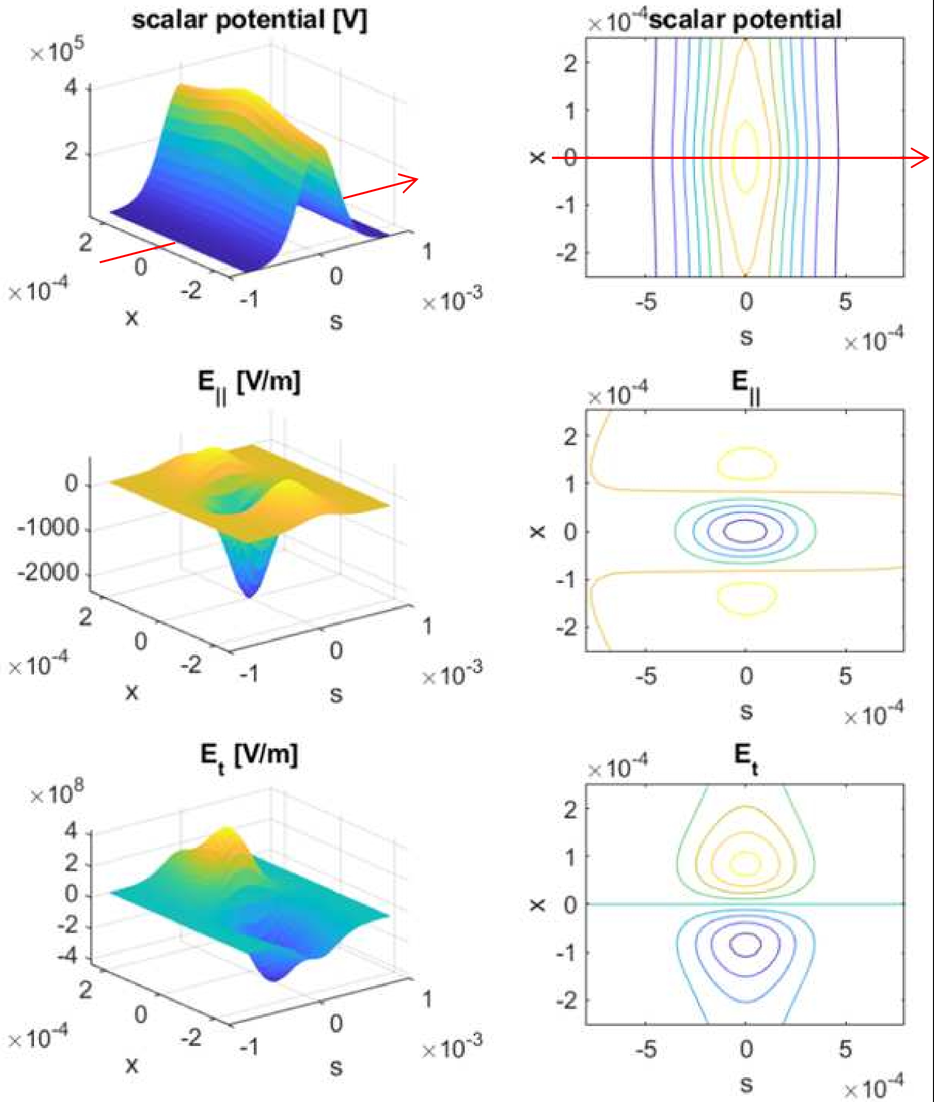
2) perturbation tracking with interpolated field

$\rightarrow$ beam dynamics properties

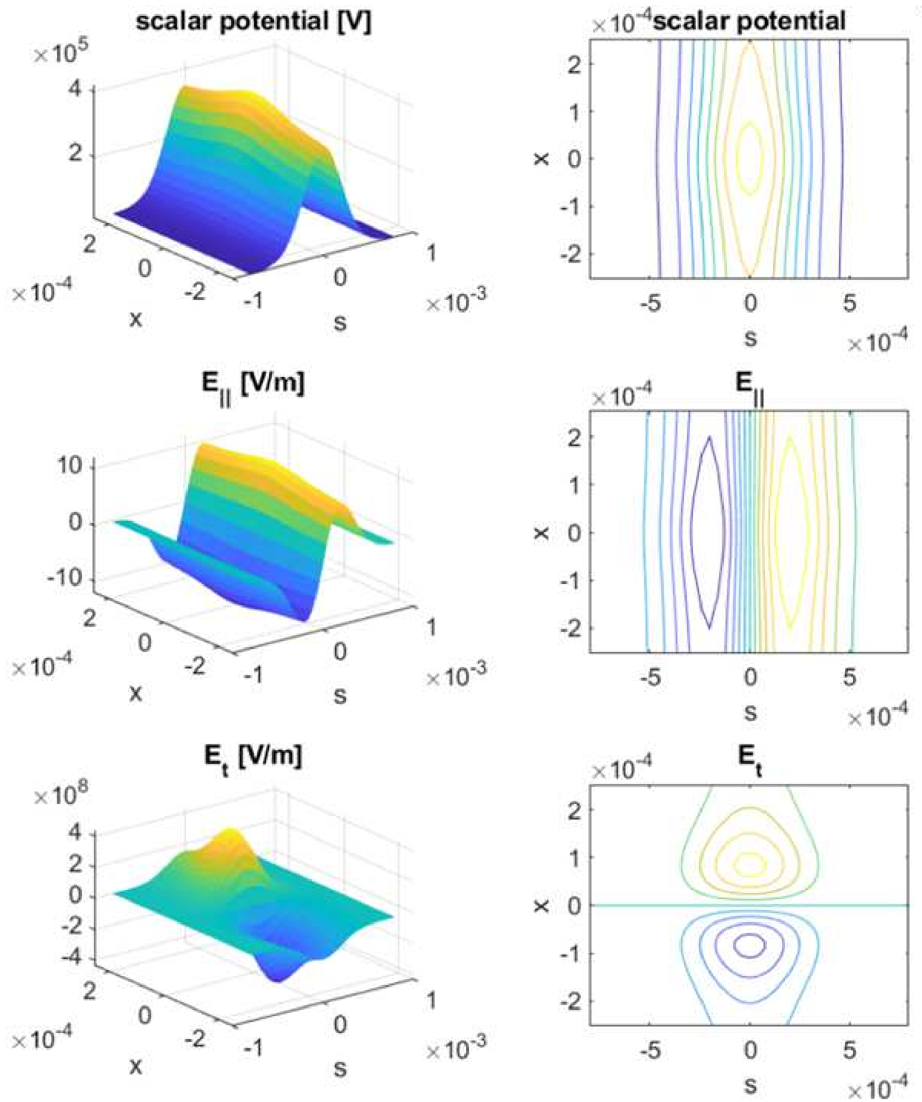
however, I am only showing one few curves for one example

benchmark BC @ 5 GeV: scalar potential and E-field **before** 1st magnet, no vertical offset

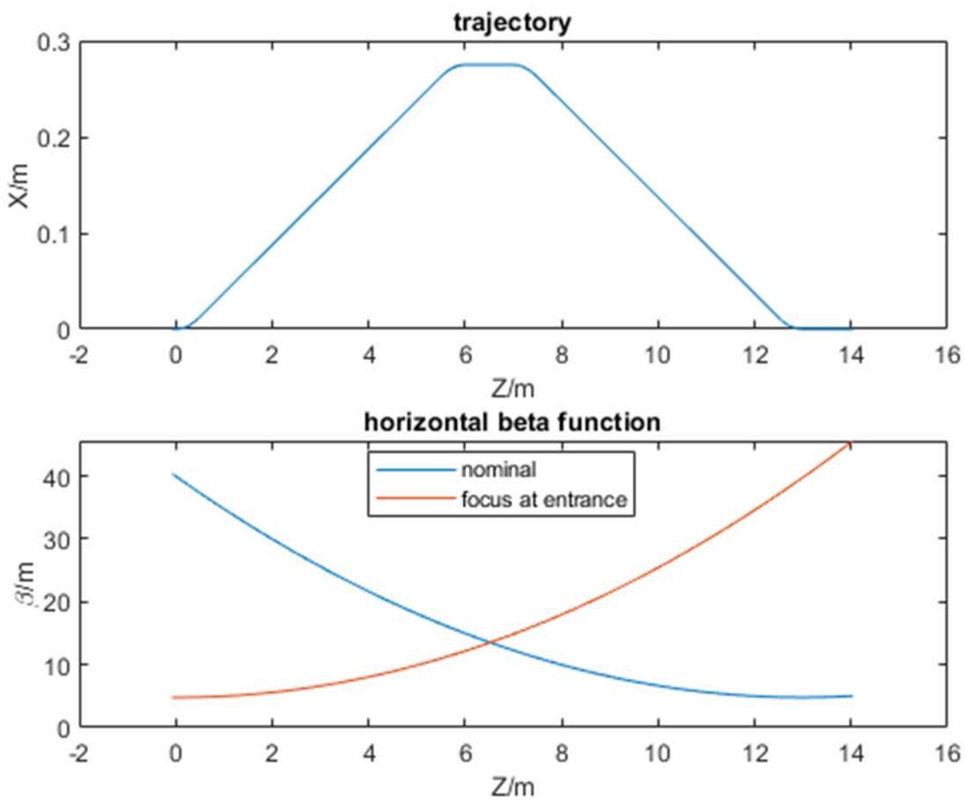
complete calculation



collective uniform motion approximation



horizontal optics: waist is at exit of BC $\rightarrow$ beam is convergent at entrance horizontal optics:

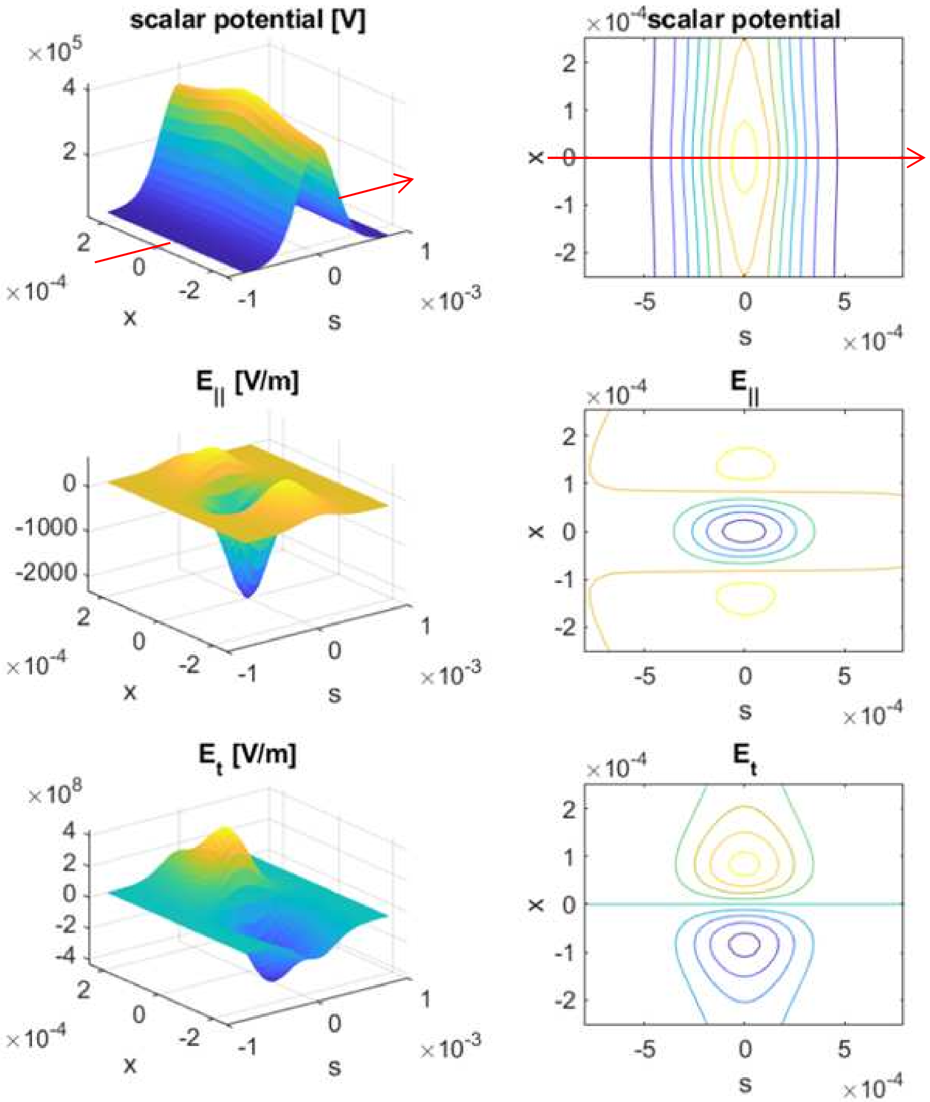


waist at entrance

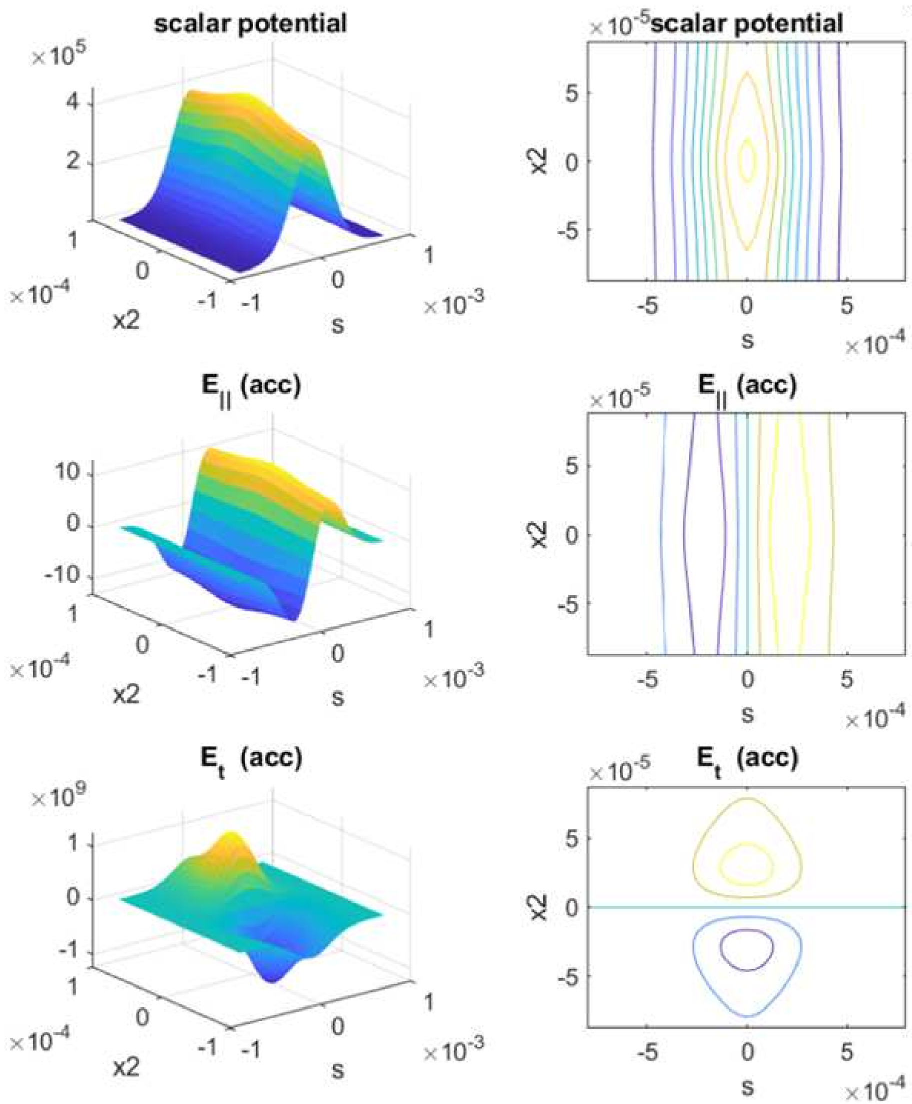
waist at exit

complete calculation: **different horizontal optics**

waist at exit



waist at entrance

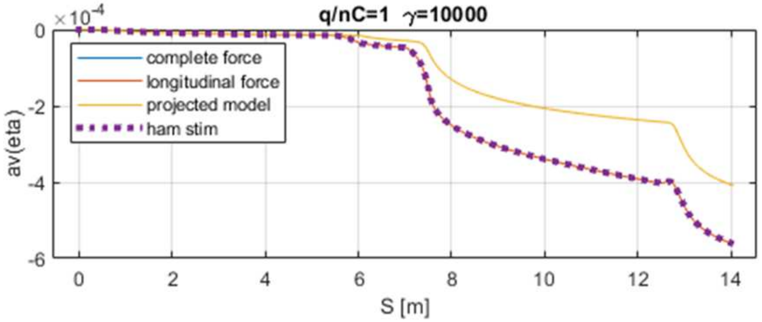


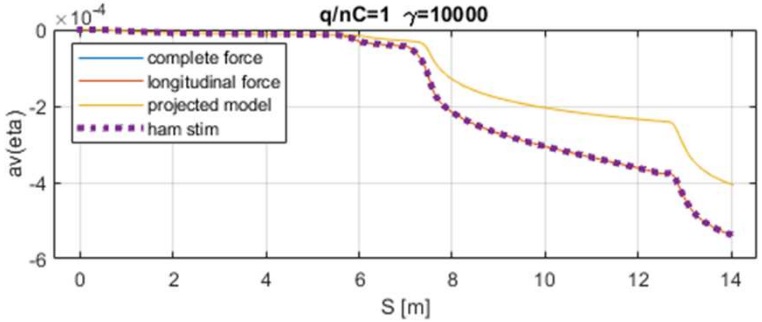
frustrating: the neglected vertical optics can cause similar effects

full beam parameters at exit due to self effect from "0" to S

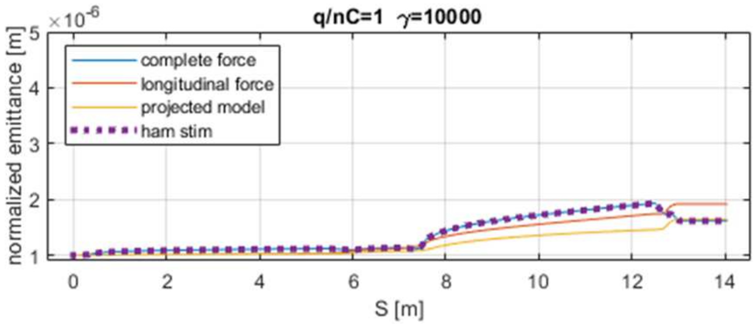
waist at exit | waist at entrance

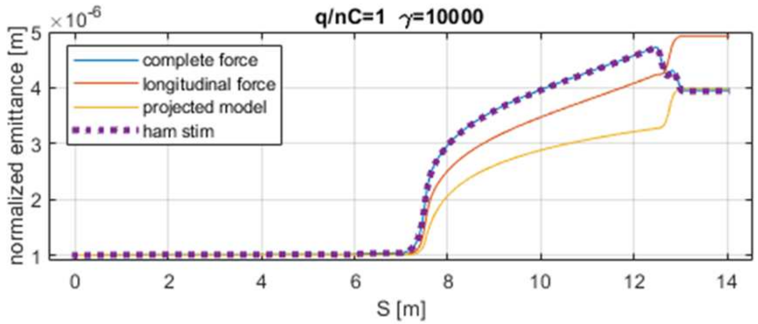
relative mean energy $\langle \eta \rangle$



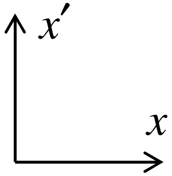


projected emittance



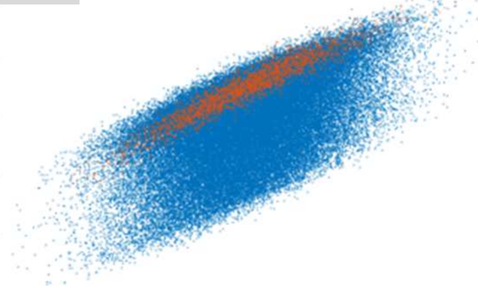


horizontal phase space at exit



full

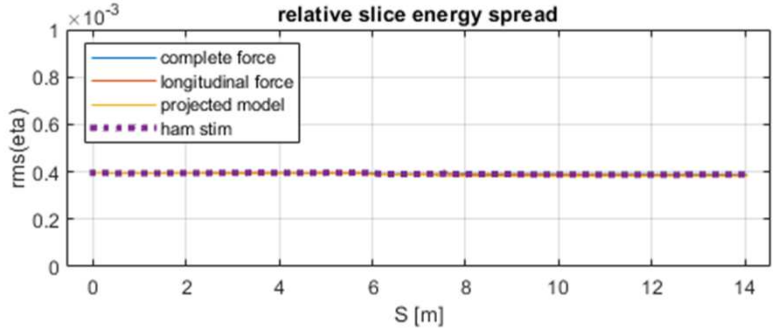
center slice

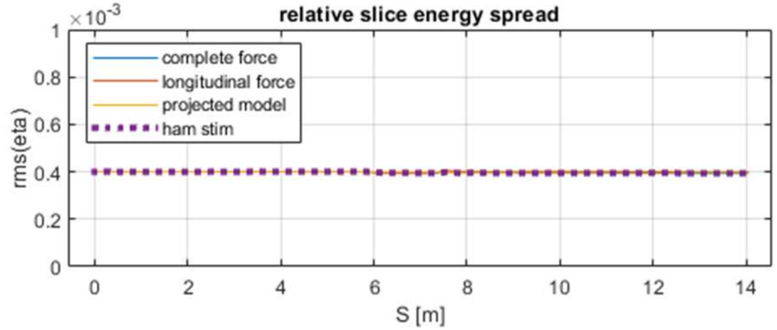


center slice parameters at exit due to self effect from “0” to S

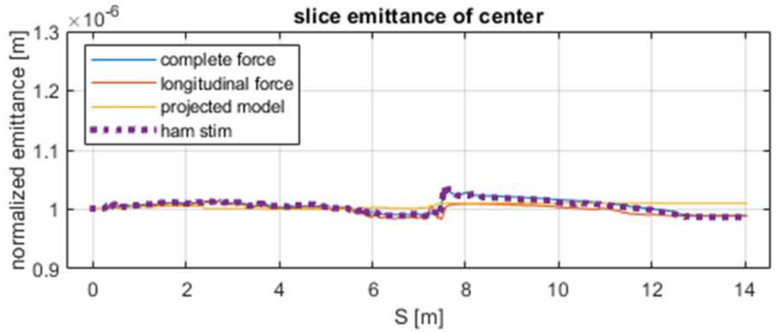
waist at exit | waist at entrance

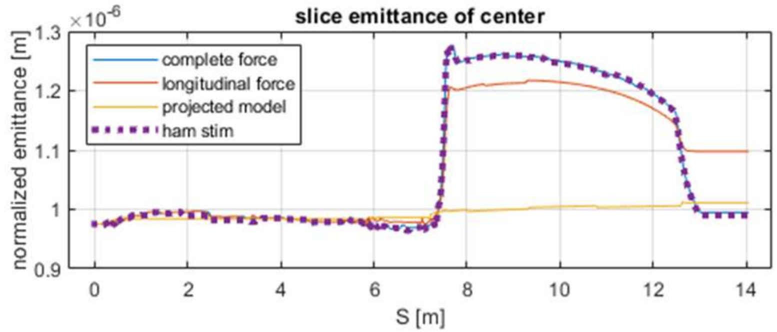
relative slice energy spread



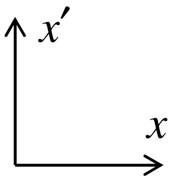


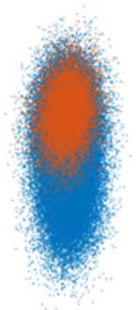
slice emittance





horizontal phase space at exit





full

center slice